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# ORLICZ SEQUENCE SPACES WITH DENUMERABLE SETS OF SYMMETRIC SEQUENCES

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ABSTRACT. We construct examples of Orlicz sequence spaces whose sets of symmetric sequences are, up to equivalence, countably infinite. This answers a question raised in [S].

## 1. Introduction

A question raised in [S] asks whether there exists an Orlicz sequence space such that the set of symmetric sequences in the space is, up to equivalence, precisely countably infinite. In this note we give an affirmative answer. In fact, for any countable ordinal $\gamma$ we construct a reflexive Orlicz sequence space whose set of symmetric sequences with the domination order is, up to equivalence, order-isomorphic to the ordinal interval $[0, \gamma]$ in the reverse order. We also show that the converse holds when the set of symmetric sequences is countable and totally ordered in the domination order. In reflexive Orlicz sequence spaces the set of symmetric sequences coincides with the set of spreading models of the space, and the problem considered here is motivated by the study of the set of spreading models. We refer to [S] and [DOS] and the references therein for more details.

## 2. Preliminaries

An *Orlicz function* $M$ is a real-valued continuous non-decreasing and convex function defined on $[0, 1]$ such that $M(0) = 0$ and $M(1) = 1$. For a given $M$, the *Orlicz sequence space* $\ell_M$ is the space of all sequences of scalars $x = (a_1, a_2, \dots)$ such that $\sum_{n=1}^{\infty} M(|a_n|/\rho) < \infty$ for some $\rho > 0$, equipped with the norm

$$\|x\| = \inf \{\rho > 0 : \sum_{n=1}^{\infty} M(|a_n|/\rho) \leq 1\}.$$

We will always assume that $M$ satisfies the $\Delta_2$-*condition* at zero (i.e., that there exists $C > 0$ such that $M(2t) \leq CM(t)$ for all $0 \leq t \leq 1/2$). Then the unit vectors form a normalized symmetric basis for $\ell_M$. If $N$ also satisfies the $\Delta_2$-condition at zero then $M$ *dominates* $N$, denoted by $N \leq M$, if there exists a constant $C > 0$ such that $N(t) \leq CM(t)$ for all $0 \leq t \leq 1$. For $M$ and $N$ satisfying the $\Delta_2$-condition at zero $N \leq M$ if and only if the unit vector basis of $\ell_M$ dominates that of $\ell_N$, that is, there exists a constant $C \geq 1$

such that for every finite sequence $(a_i)$ of scalars,

$$\|\sum_i a_i e_i\|_{\ell_N} \le C\|\sum_i a_i e_i\|_{\ell_M}.$$

We say that two such functions $M$ and $N$ are *equivalent* if $M \le N$ and $N \le M$. Thus $M$ is equivalent to $N$ if and only if the unit vector bases of $\ell_M$ and of $\ell_N$ are equivalent.

If $M$ satisfies the $\Delta_2$-condition at zero then an Orlicz sequence space $\ell_N$ is isomorphic to a subspace of $\ell_M$ if and only if $N$ is equivalent to some function in $C_{M,1}$, where $C_{M,1}$ is the norm-closed convex hull in $C[0,1]$ of the set

$$E_{M,1} = \overline{\Big\{\frac{M(\lambda t)}{M(\lambda)};\ 0 < \lambda < 1\Big\}}. \tag{2.1}$$

See [LT, Lemma 4.a.6, remark (p. 141) and Theorem 4.a.8] for this result. Let $(x_i)$ be a normalized symmetric (or even subsymmetric) sequence in $\ell_M$. Then $(x_i)$ is equivalent to the unit vector basis of $\ell_N$, where $N \in C_{M,1}$ ([LT], Proposition 4.a.7 and Theorem 4.a.8). Thus there is a one-to-one correspondence between the set of symmetric sequences in $\ell_M$ and the Orlicz functions $N$ in $C_{M,1}$. Moreover, these two sets are *order-isomorphic*. That is, if $(x_i)$ and $(y_i)$ are two normalized symmetric sequences in $\ell_M$ and $N_1$ and $N_2$ are the corresponding functions in $C_{M,1}$ respectively, then $(y_i)$ dominates $(x_i)$ if and only if $N_1 \le N_2$.

We will use the following method of representing Orlicz functions by sequences of zeros and ones, introduced by Lindenstrauss and Tzafriri [LT, p. 161].

Fix $0 < \tau < 1$ and $1 < r < p < \infty$. For every sequence of zeros and ones, $\eta = (\eta(n))_{n=1}^{\infty}$ (i.e. $\eta(n) \in \{0,1\}$ for all $n$), let $M_\eta$ be the piecewise linear function defined on $[0,1]$ satisfying $M_\eta(0) = 0$, $M_\eta(1) = 1$, and

$$M_\eta(\tau^k) = \tau^{rk+(p-r)\sum_{n=1}^{k}\eta(n)}, \qquad k = 1, 2, \ldots.$$

In our construction we will use the following particular class of such spaces as considered in [DOS, Lemma 2.3].

**Lemma 2.1** (DOS). *Let $1 < r < p < \infty$, and $0 < \tau < 1$. Let $(n_k) \subset \{1, 2, 2^2, \ldots\}$ with $n_1 = 1$ and put*

$$\rho(i) = \begin{cases} 0 & \text{if } i = n_k \\ 1 & \text{otherwise.} \end{cases}$$

*Let $M := M_\rho$ be the corresponding Orlicz function. Then $\ell_M$ is reflexive and every symmetric sequence in $\ell_M$ is equivalent to either the unit vector basis of $\ell_M$ or to the unit vector basis of $\ell_p$.*

## 3. The Construction

**Theorem 3.1.** *For every countable ordinal $\gamma \geq 0$ there exists a reflexive Orlicz sequence space $\ell_M$ such that the set of normalized symmetric sequences in $\ell_M$ is order-isomorphic to $[0, \gamma]$ with the reverse order.*

*Proof.* The first part of the construction bears similarities to the one in [DOS, Theorem 3.1]. For every $\alpha < \gamma$ we construct a sequence $\rho_\alpha$ of the form given in Lemma 2.1 so that the collection of corresponding Orlicz functions $(M_{\rho_\alpha})_{\alpha<\gamma}$ (for the same $\tau$, $r$ and $p$) is order-isomorphic to $[0, \gamma] \setminus \{\gamma\}$ with the reverse order. Then we patch the $\rho_\alpha$ sequences together into one sequence $\phi$ in such a way that $C_{M_\phi,1}$ of the corresponding Orlicz function $M_\phi$ (again with the same $\tau$, $r$ and $p$) will consist, up to equivalence, only of the collection $\{M_\phi\} \cup \{M_{\rho_\alpha}\}_{\alpha<\gamma} \cup \{t^p\}$. Thus $\ell_{M_\phi}$ will be the desired Orlicz sequence space.

The $\rho_\alpha$ sequences will be constructed simultaneously on a finite interval at a time using a variation on what is called the *$\varepsilon$-domination procedure* in [DOS], which we recall now for completeness.

Let $(\rho_j)$ be a collection of sequences to be constructed, and let $A \subset \mathbb{N}$ and $\varepsilon > 0$. Put $\rho_j(1) = 0$ for all $j$, and suppose that all the sequences are already defined on $[1, N]$.

The $(\varepsilon, A)$-domination procedure extends the definition of the $\rho_j$'s to an initial segment $[1, N_1]$ for some $N_1 > N$.

Suppose that both $A$ and $\mathbb{N} \setminus A$ are non-empty. Choose a sufficiently large (just how large is specified below) integer $m > N$. For all $k \in \mathbb{N} \setminus A$ place 0's on the coordinates from $[N+1, m] \cap \{2^n : n = 0, 1, 2, \ldots\}$ of the $\rho_k$'s (while the rest of the coordinates of the interval are filled with 1's), and for all $j \in A$ place 1's on *all* the coordinates from $[N + 1, m]$ of the $\rho_j$'s , where $m$ is chosen so that

$$\sum_{i=1}^{m} \rho_j(i) - \sum_{i=1}^{m} \rho_k(i)$$

is sufficiently large to ensure that

$$\frac{M_{\rho_j}(\tau^m)}{M_{\rho_k}(\tau^m)} < \varepsilon, \text{ for all } j \in A,\ k \in \mathbb{N} \setminus A.$$

We now pass to the main construction. Let $\gamma$ be a countable ordinal. For simplicity we shall assume that $\gamma$ is a limit ordinal (the argument in the successor ordinal case is similar). Let $L = [0, \gamma] \setminus \{\gamma\}$ with the reverse order. Note that every subset of $L$ has a *maximum* element in this order. Let $(\alpha_j)_{j=1}^{\infty}$ be an enumeration of $L$.

For every $j \in \mathbb{N}$ and every $\varepsilon_k = 2^{-k}$, $k = 1, 2, \ldots$, we carry out an $(\varepsilon_k, A_j)$-domination procedure for $A_j = \{i \in \mathbb{N} : \alpha_i < \alpha_j \text{ in } L \text{ order}\}$ as described above. Note that $A_j$ and $\mathbb{N} \setminus A_j$ are both nonempty. We shall call $\alpha_j$ the *dominating node* and we shall say that $\alpha_i$

is $\varepsilon$-*dominated* by $\alpha_j$ if $i \in A_j$. Since there are countably many choices we can enumerate some order in which to carry out all $(\varepsilon_k, A_j)$-dominations.

At the end we obtain sequences $\rho_j$ for each $\alpha_j \in L$ with the following easily checked properties:

(i) Each $\rho_j$ begins with 0;

(ii) The zeros occur only on coordinates $(2^k)$;

(iii) $\alpha_i \geq \alpha_j$ (in the order of $L$) if and only if $\rho_i(n) \leq \rho_j(n)$ for all $n$ (we say that $\rho_j$ dominates $\rho_i$ *pointwise*). Thus in this case $M_{\rho_j} \leq M_{\rho_i}$;

(iv) $L$ is order-isomorphic to the collection $(M_{\rho_j})$.

Now let $\phi$ be the sequence obtained by writing out the initial segments of the $\rho_j$'s with extra 1's so that each initial segment of length $n$ is followed by $n$ 1's before the next initial segment starts:

$$\phi = (\rho[1], 1, \rho_1[2], 1, 1, \rho_2[2], 1, 1, \rho_1[3], 1, 1, 1, \rho_2[3], 1, 1, 1, \rho_3[3], 1, 1, 1, \rho_1[4], \ldots),$$

where $\rho[n]$ denotes the initial segment of $\rho$ of size $n$.

We will show that $M := M_\phi$ is the desired Orlicz function. Recall that $N \in E_{M_\rho,1}$ if and only if $N = N_\psi$ where $\psi$ is a pointwise limit of $(T^m\phi)_{m\in\mathbb{N}}$ [LT, p. 161]. From the form of $\phi$ it is clear that the $M_{\rho_j}$ functions and $t^p$, which is represented by the sequence $(1, 1, 1, \ldots)$, belong to $E_{M,1}$, and, moreover, that $M_\phi$ strictly dominates each $M_{\rho_j}$.

We show in the next lemma that, together with $M_\phi$, these are the only elements of $E_{M,1}$ up to equivalence.

For $n \geq 0$ by $T^n\rho$ denote the sequence obtained by shifting $\rho$ to the left $n$ coordinates, and for $n < 0$ by $T^n\rho$ denote the sequence $\rho$ with $n$ 1's put at the beginning.

One easily verifies from the definition of $\rho_\alpha := \rho_j$ that $M_{\rho_\alpha}(t) \geq M_{T^m(\rho_\alpha)}(t)$ for all $\alpha = \alpha_j \in L$, $m \in \mathbb{Z}$, and $t \in [0, 1]$. This fact will be used in the proof of our main result.

**Lemma 3.2.** *The pointwise limits of $(T^m\phi)_{m\in\mathbb{N}}$ are of the form*

*(a) all sequences $T^m\rho_j$ for $\alpha_j \in L$ and $m \in \mathbb{Z}$, or*

*(b) all sequences with at most one term equal to zero and all other terms equal to one.*

*Proof of Lemma 3.2.* Let $\psi$ be such a limit. If $\psi$ is not of type (b) then there is a fixed $m_0 \in \mathbb{Z}$ so that $\psi$ is a pointwise limit of a sequence of sequences of the form $(T^{m_0}\rho_{j_k})_{k\geq 1}$. For each $i \in \mathbb{N}$, let $\alpha_{j(i)}$ be the greatest element in $L$ such that $T^{m_0}\rho_{j(i)}$ agrees with $\psi$ on the first $i$ coordinates. (Every subset of $L$ has a maximum, so $\alpha_{j(i)}$ exists.) Clearly, $\alpha_{j(1)} \geq \alpha_{j(2)} \geq \alpha_{j(3)} \geq \ldots$. Let $\alpha = \lim_i \alpha_{j(i)}$. The limit exists in $L \cup \{\gamma\}$ since $L \cup \{\gamma\}$ is an ordinal in the reverse order.

If $\alpha > \gamma$, then we claim that $\psi = T^{m_0}\rho_\alpha$. Indeed, clearly $T^{m_0}\rho_\alpha \geq \psi$ pointwise since $\rho_\alpha \geq \rho_{j(i)}$ pointwise for all $i$ (by construction). But consider a subinterval of $\mathbb{N}$ on which $\alpha$

is $\varepsilon$-dominated by some $\beta$ with $\beta > \alpha$. Since $\alpha_{j(i)} \searrow \alpha$ we have $\beta > \alpha_{j(i)}$ for all sufficiently large $i$. Hence $\alpha_{j(i)}$ is also $\varepsilon$-dominated by $\beta$ for all sufficiently large $i$. This proves that $\psi \geq T^{m_0}\rho_\alpha$ pointwise.

If $\alpha = \gamma$ then a similar argument shows that $\psi$ agrees with an $\varepsilon$-dominated sequence on every subinterval of $\mathbb{N}$ on which an $\varepsilon$-domination takes place, and hence $\psi$ is of type (b). $\square$

It remains to show that every function in $C_{M,1}$ is equivalent to $M_\phi$ or to one of the $M_{\rho_j}$'s or to $t^p$.

Let $M_\sigma \in C_{M,1}$. Then by Lemma 3.2 $M_\sigma$ is equivalent either to $M_\phi$ or to a uniform limit of a sequence $(M_{\sigma_n})$ of finite convex combinations of functions in $E_{M,1}$ of the form

$$M_{\sigma_n} = \sum_{m\in\mathbb{Z},\alpha\geq\gamma} \lambda^n_{m,\alpha} M_{T^m\rho_\alpha},$$

where the positive 'mass' coefficients $\lambda^n_{m,\alpha}$ sum to 1 for each $n$. Here $\rho_\gamma := (1,1,1,\dots)$ (so $M_{\rho_\gamma}$ is equivalent to $t^p$) represents the minimum symmetric sequence in the domination order.

Define the following function $\xi$ on $L$:

$$\xi(\alpha) = \lim_{m\to\infty} \limsup_{n\to\infty} \sum_{\beta\geq\alpha,|j|\leq m} \lambda^n_{j,\beta}.$$

We now consider two cases.

In the first case, suppose that $\xi$ is not identically zero. Then there exists a greatest element $\alpha_0$ with $\xi(\alpha_0) > 0$ (since every subset of $L$ has a maximum element). This implies that there exists $m_0 \in \mathbb{N}$ with

$$\limsup_{n\to\infty} \sum_{\beta\geq\alpha_0,|j|\leq m_0} \lambda^n_{j,\beta} := \delta > 0.$$

We claim that in this case $M_\sigma$ is equivalent to $M_{\rho_{\alpha_0}}$. Clearly, $M_\sigma$ dominates $M_{\rho_{\alpha_0}}$ since $\delta > 0$ and $M_{\rho_\beta} \geq M_{\rho_\alpha}$ whenever $\beta \geq \alpha$. Consider a fixed initial segment of $\mathbb{N}$ on which $\alpha_0$ is $\varepsilon$-dominated at least once. Then the collection of dominating nodes which $\varepsilon$-dominate $\alpha_0$ on this initial segment is finite and non-empty. So there is a least (in the ordering of $L$) dominating node $\beta_0 > \alpha_0$ from this collection.

Thus $\xi(\beta_0) = 0$ by the definition of $\alpha_0$. This means that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \sum_{\beta\geq\beta_0,|j|\leq m} \lambda^n_{j,\beta} = 0.$$

Note that if $|m| > C$, where $C$ depends only on the fixed initial segment of $\mathbb{N}$, then for all $\alpha \in L$, $T^m\rho_\alpha$ will contain at most one zero in that initial segment. But the sequences $T^m\rho_\alpha$ for $\alpha \geq \beta_0$ and $m \leq C$ which occur in the definition of $M_{\sigma_n}$ are assigned a 'total mass' which tends to zero as $n \to \infty$, and hence their contribution vanishes in the limit. On the other hand, if $\alpha < \beta_0$, then on this initial segment $\alpha$ is $\varepsilon$-dominated whenever $\alpha_0$

is $\varepsilon$-dominated, and hence $\rho_{\alpha_0}$ is dominated pointwise on this initial segment by $\rho_\alpha$. Recall that $M_\alpha(t) \geq M_{T^m(\alpha)}(t)$ for all $\alpha \in L$, $m \in \mathbb{Z}$, and $t \in [0,1]$. Taking the limit as $n \to \infty$ it follows that $M_\sigma \leq M_{\rho_{\alpha_0}}$.

In the second case, suppose that $\xi(\alpha) = 0$ for all $\alpha$. This easily implies that $M_\sigma$ is equivalent to the minimum element $M_{\rho(\gamma)}$ (which is equivalent to $t^p$).

Thus, we have proved that $\{\ell_{M_\phi}\} \cup \{\ell_{M_{\rho_j}}\}_{j=1}^{\infty} \cup \{\ell_p\}$ is the set of symmetric sequences in $\ell_{M_\phi}$ up to equivalence. In the domination order, $\ell_{M_\phi}$ is the maximum element and $\ell_p$ is the minimum element, and hence the set is order-isomorphic to the ordinal sum $1 + \gamma^+$ with the reverse order, where $\gamma^+$ denotes the successor ordinal to $\gamma$, which in turn is isomorphic to the ordinal interval $[0, \gamma]$ with the reverse order. □

**Corollary 3.3.** *Suppose that $L$ is a countable totally ordered set. Then there exists a reflexive Orlicz space $\ell_M$ such that the collection of symmetric sequences in $\ell_M$ with the domination order is order-isomorphic to $L$ if and only if $L$ is order-isomorphic to an ordinal interval $[0, \gamma]$ in the reverse order for some countable ordinal $\gamma$.*

*Proof.* Sufficiency follows from our main result. Conversely, if $L$ is the set of symmetric sequences up to equivalence in $\ell_M$, where $\ell_M$ is reflexive, then $L$ coincides with the set of spreading models of $\ell_M$ up to equivalence [S], and if $L$ is countable and totally ordered, then $L$ is well-ordered in the reverse order [DOS, Theorem 3.7], and moreover $L$ has a minimum element [S]. Hence $L$ is order-isomorphic to $[0, \gamma]$ in the reverse order for some countable ordinal $\gamma$. □

## References


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